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Multipole moments in general relativity

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Abstract. The Bianchi identities are analysed to first order. There are two cases to consider. First, we examine the case in which the sources for the gravitational field are ignored, and obtain the solutions of the resulting homogeneous equations for the frame components of the Weyl spinor. Second, we include the sources and obtain solutions of the wave-like equation which occurs. The two cases are compared, especially with reference to the multipole moments which arise from these analyses. The moments arising from the solutions of the homogeneous case are due to contributions from terms with different dimensions. The moments of the inhomogeneous case are associated with a single dimension only, and are used together with certain integral expressions involving the Bondi–Sachs news function to determine the main contributions to the energy momentum losses of gravitationally radiating isolated sources.

1. Introduction

The extreme difficulties inherent in the nonlinearity of the equations of general relativity force us, in the discussion of gravitational radiation, to adopt some form of approximation procedure. The procedure in which successive approximations about a flat space–time background are taken has been considered by several authors, notably (from the point of view of this paper) Janis and Newman (1965) who discussed the first-order (linearised) approximation, and Torrence and Janis (1967), Couch *et al* (1968) who dealt with the second-order approximations. In the first-order approximation the equations which occur are linear first-order partial differential equations for certain 'frame' components of the Weyl tensor. In the above papers the free field solution is considered, whence terms representing the sources for the radiation field are ignored; the linear equations are homogeneous in the frame components. In this paper we extend the discussion of the linear approximation to the inhomogeneous case (in which the source terms are included).

The analysis of the homogeneous case yields certain functions (denoted by $\overset{(l)}{h}_q$ in this paper) called multipole moments, which arise via coefficients of the spin weighted spherical harmonics (which occur when null spherical polar coordinates are used). The way in which the analysis is performed in the inhomogeneous case leads in a natural way to a definition of multipole moment (denoted by $h_{\alpha\beta|c_1\ldots c_n}$) based on dimensional behaviour rather than angular behaviour. It turns out that the $\overset{(l)}{h}_q$ are due to contributions from terms with different dimensions, a behaviour which is described by the term 'dimensional mixing'. Even for the simplest systems this mixing is infinite; an infinite number of terms with different dimensions contribute to any $\overset{(l)}{h}_q$. This is markedly different from the $h_{\alpha\beta|c_1\ldots c_n}$, which are associated with a single dimension.

As far as establishing general features of gravitational radiation the source dependent moments $h_{\alpha\beta|c_1\ldots c_n}$ are not of great use, leading as they do to a cumbersome analysis. Features like wave tails, and the *existence* of mass and momentum losses from radiating bodies can be (and have been) perfectly well established using the $\overset{(l)}{h}_q$. However, by its very nature, the source dependent definition of moment is the one which should be employed whenever specific calculations related to given source configurations are required. This holds in particular for energy momentum losses. If the pseudotensor of energy is used in these calculations they turn out to be long and involved. This difficulty can be removed by using the Bondi–Sachs news function instead. This quantity will be used in this paper, and will be combined with the source dependent definition of moment to obtain the main contributions for energy momentum losses. The expressions are simple, and completely general.

The sources are assumed spatially compact i.e. of finite extent. We shall also suppose that the moments are constant except for finite retarded time intervals, a condition described by saying 'the source is in motion only for finite time intervals'. This phraseology, it should be clearly understood, refers only to linear theory. If nonlinear approximations are taken into account then the source will in general recoil during the period in which the moments are non-constant. This recoil continues after the moments have once again assumed constant values.

The paper is arranged as follows. In § 2 the solution for the homogeneous case is derived. The results are not new but are obtained for completeness, and because they will be required for comparison with the results of the inhomogeneous case, which are obtained in § 3. Dimensional mixing is also discussed in some detail in this section. In § 4 the formulae for energy momentum losses are derived and subsequently applied, in § 5, to a particular example—that of the rotating rod. This example is used merely to illustrate the previous theory; the results, especially for the energy loss of the rod, are well known.

2. The Bianchi identities and the free field solution

In the following and throughout the paper Latin capitals will be used for spinor indices and will take the values 0, 1, and Greek letters will be used for tensor indices and will take the values 0, 1, 2, 3. The metric has signature −2. Familiarity with spinor calculus is assumed in both this and the next section. The notation follows that of Newman and Penrose (1962), §§ 3, 4 (see also Penrose 1960, Pirani 1964).

In general space–time the Bianchi identities in spinor form are

$$\partial^{D}{}_{G'}\Psi_{ABCD} = \partial_{(C}{}^{H'}\Phi_{AB)G'H'} \tag{2.1a}$$

$$\partial^{AG'}\Phi_{ABG'H'} = -3\partial_{BH'}\Lambda \tag{2.1b}$$

where the spinors Ψ_{ABCD}, $\Phi_{ABA'B'}$ and Λ correspond to the Weyl tensor, trace free part of the Ricci tensor, and scalar curvature respectively. The $\partial_{AB'}$ are covariant differentiation operators. There are no identities independent of these.

In empty space we can put $\Phi_{ABA'B'} = \Lambda = 0$ whence (2.1) reduce to

$$\partial^{D}{}_{G'}\Psi_{ABCD} = 0. \tag{2.2}$$

Let us now linearise (2.2) i.e. let us work to first order in the expansion about the flat space–time background, whose metric in null spherical polar coordinates u, r, θ, ϕ is given by

$$ds^2 = du^2 + 2\,du\,dr - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{2.3}$$

from which a spin dyad o^A, ι^A is obtained which takes the form

$$o^A = 2^{1/4}\begin{Bmatrix}\cos\frac{1}{2}\theta\,e^{-i\phi/2}\\ \sin\frac{1}{2}\theta\,e^{i\phi/2}\end{Bmatrix} \qquad \iota^A = 2^{-1/4}\begin{Bmatrix}-\sin\frac{1}{2}\theta\,e^{-i\phi/2}\\ \cos\frac{1}{2}\theta\,e^{i\phi/2}\end{Bmatrix}. \tag{2.4}$$

Replacing the covariant operators $\partial_{AB'}$ by the partial differentiation operators defined with respect to the metric (2.3) and taking dyad components we obtain

$$\begin{aligned} &o^A\bar{o}^{A'}\partial^{\text{flat}}_{AA'} = \frac{\partial}{\partial r} \qquad o^A\bar{\iota}^{A'}\partial^{\text{flat}}_{AA'} = \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\ &\iota^A\bar{\iota}^{A'}\partial^{\text{flat}}_{AA'} = \partial/\partial u - \tfrac{1}{2}\partial/\partial r \end{aligned} \tag{2.5}$$

since $\partial^{\text{flat}}_{AA'} = \sigma^{\mu}{}_{AA'}\,\partial/\partial x^{\mu}$, where $\sigma^{\mu}{}_{AA'}$ are the symbols providing a translation from tensor indices to spinor indices (they may be taken as $2^{-1/2}\times$ the unit matrix and the complex conjugates of the Pauli matrices), and where $x^{\mu} = (t, x, y, z) = (u + r, r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ are Minkowski coordinates. Taking dyad components of (2.2) and using (2.4), (2.5) we readily find that (2.2) reduce to the following set of eight equations

$$\left(\partial_r + \frac{5-k}{r}\right)\psi_k + \frac{1}{\sqrt{2}r}\bar{\eth}\psi_{k-1} = 0 \tag{2.6}$$

and $\qquad\qquad (k = 1, 2, 3, 4)$

$$\dot{\psi}_{k-1} - \frac{1}{2}\left(\partial_r + \frac{k}{r}\right)\psi_{k-1} + \frac{1}{\sqrt{2}r}\eth\psi_k = 0 \tag{2.7}$$

where $\partial_r \equiv \partial/\partial r$, $\cdot \equiv \partial/\partial u$, and $\eth$, $\bar{\eth}$ are spin weight raising and lowering operators (see the appendix)

$$\begin{aligned} \eth\psi_k &= -\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + (k-2)\cot\theta\right)\psi_k\\ \bar{\eth}\psi_k &= -\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + (2-k)\cot\theta\right)\psi_k \end{aligned} \tag{2.8}$$

and where ψ_k $(k = 0, \ldots, 4)$ are the dyad components of the spinor Ψ_{ABCD}. At this stage an assumption on the behaviour of ψ_0 at large r must be made in order to ensure that the space–time is asymptotically flat. We assume that†

$$\psi_0 = O(r^{-5}). \tag{2.9}$$

With this (2.6) are immediately integrated to give

$$\psi_k = \frac{\psi_k^0}{r^{5-k}} - \frac{1}{\sqrt{2}r^{5-k}}\int_{\infty}^{r} r'^{(4-k)}\bar{\eth}\psi_{k-1}\,dr' \tag{2.10}$$

† $f(u, r, \theta, \phi) = O(g(r))$ means that $|f(u, r, \theta, \phi)| < g(r)F(u, \theta, \phi)$ for some F independent of r and for all sufficiently large r.

the ψ_k^0 being functions of integration independent of r. The $k=1$ member of (2.7) becomes

$$\dot\psi_0-\frac{1}{2}\left(\partial_r+\frac{1}{r}\right)\psi_0-\frac{1}{2r^5}\int_\infty^r r'^3\eth\bar\eth\psi_0\,\mathrm{d}r'+\frac{1}{\sqrt{2}r^5}\eth\psi_1^0=0. \tag{2.11}$$

The component ψ_0 is a spin weight 2 quantity; hence it can be expanded in spin weight 2 spherical harmonics (see the appendix)

$$\psi_0=\sum_{l=2}^{\infty}\sum_{q=-l}^{l}A_{lq}(u,r)\eth^2(P_l^{|q|}\,\mathrm{e}^{\mathrm{i}q\phi}). \tag{2.12}$$

Let us now seek a solution for A_{lq} throughout the space–time exterior to the sources for the field in the form of a finite series in inverse powers of r. Noting the asymptotic behaviour (2.9) we try

$$A_{lq}(u,r)=\sum_{n=0}^{p}\frac{a_{lq}^n(u)}{r^{n+5}} \tag{2.13}$$

with p finite. Substituting this into (2.11) and considering separately the coefficients of the different powers of r we obtain

$$\dot a_{lq}^n(u)=\left(\frac{(l-1)(l+2)-n(n+3)}{2n}\right)a_{lq}^{n-1}(u)\qquad(n=1,\ldots,p) \tag{2.14a}$$

and

$$\left(\frac{(l-1)(l+2)-(p+1)(p+4)}{2(p+1)}\right)a_{lq}^p(u)=0. \tag{2.14b}$$

The equation involving $\dot a_{lq}^0$ is not important here. We see from (2.14) that p can be chosen to be $(l-2)$, for if $p>l-2$ all $a_{lq}^n(u)$ automatically vanish for $n\geq l-1$. Further, once $a_{lq}^{l-2}(u)$ is known, so also are all other $a_{lq}^n(u)$ $(n=0,1,\ldots,l-3)$. Hence by specifying $a_{lq}^{l-2}(u)$ we obtain a solution for A_{lq} of the form (2.13), with $p=l-2$ and $\dot a_{lq}^n(u)$ $(n\leq l-2)$ satisfying (2.14a) (with (2.14b) automatically satisfied). Now, since we are considering linearised theory, all field quantities can be considered as being of first order in a parameter m characterising the mass of the radiating system. From the above remarks we accordingly set

$$a_{lq}^{l-2}(u)=m\overset{(l)}{h}_q(u) \tag{2.15}$$

the functions $\overset{(l)}{h}_q(u)$ being independent of any mass dimension. These $\overset{(l)}{h}_q(u)$ are defined as the 'multipole moments' of the source. The solution for the field corresponding to $\Sigma_{q=-l}^{l}\overset{(l)}{h}_q(u)$ is called the 2^l-pole (or $(1l)$) solution 'related to the free field'.

Using (2.12)–(2.15) we find that the $(1l)$ solutions for ψ_0 (denoted $\overset{(l)}{\psi}_0$) are

$$\begin{aligned}&\overset{(0)}{\psi}_0=\overset{(1)}{\psi}_0=0\\&\overset{(l)}{\psi}_0=\sum_{n=1}^{l-1}\sum_{q=-l}^{l}\frac{{}^lC_n\overset{(l)}{h}{}_q^{(l-n-1)}}{r^{n+4}}\eth^2(P_l^{|q|}\,\mathrm{e}^{\mathrm{i}q\phi})\qquad(l\geq2)\end{aligned} \tag{2.16}$$

where

$$\overset{l}{c}_n=\frac{2^{l-n-1}(l+n+1)!(l-2)!}{(2l)!(l-n-1)!(n-1)!} \tag{2.17a}$$

$$\overset{(l)}{h}_q{}^{(k)}\equiv\frac{\mathrm{d}^k}{\mathrm{d}u^k}\overset{(l)}{h}_q. \tag{2.17b}$$

The remainder of the Weyl field i.e. the components ψ_i $(i=1,2,3,4)$ can now be determined via further consideration of equations (2.6) and (2.7), and can be simplified slightly by use of Birkhoff's theorem and certain coordinate transformations. The details of these calculations are not of any great value and are omitted. Of interest is the solution for ψ_4

$$\overset{(0)}{\psi}_4=\overset{(1)}{\psi}_4=0$$

$$\overset{(l)}{\psi}_4=\bigg[\sum_{q=-l}^{l}\bigg(\frac{4(l-2)!}{(l+2)!}\frac{\overset{l}{c}_1\overset{(l)}{h}_q{}^{(l+2)}}{r}+\frac{2\overset{l}{c}_1\overset{(l)}{h}_q{}^{(l+1)}}{l(l+1)r^2}+\frac{\overset{l}{c}_1\overset{(l)}{h}_q{}^{(l)}}{2r^3}+\frac{l(l+1)\overset{l}{c}_1\overset{(l)}{h}_q{}^{(l-1)}}{12r^4}\bigg)$$
$$+\frac{(l+2)!}{4(l-2)!}\sum_{n=1}^{l-1}\sum_{q=-l}^{l}\frac{\overset{l}{c}_n\overset{(l)}{h}_q{}^{(l-n-1)}}{n(n+1)(n+2)(n+3)r^{n+4}}\bigg]\bar{\eth}^2(P_l^{|q|}\,\mathrm{e}^{\mathrm{i}q\phi})\qquad(l\geqslant 2). \tag{2.18}$$

ψ_1, ψ_2 and ψ_3 do not contribute to any further discussion.

3. The Bianchi identities in non-empty space

We now consider (2.1) in full generality i.e. $\Phi_{ABA'B'}\neq 0\neq\Lambda$. Applying the operator $\partial_{FG'}$ to (2.1a) gives

$$\begin{aligned}\partial_{FG'}\,\partial_D{}^{G'}\Psi_{ABC}{}^D&\equiv\tfrac{1}{2}\{\partial_{FG'}\,\partial_D{}^{G'}+\partial_{DG'}\,\partial_F{}^{G'}\}\Psi_{ABC}{}^D-\tfrac{1}{2}\Box\Psi_{ABCF}\\&=-\partial_{FG'}\,\partial_{(C}{}^{H'}\Phi_{AB)}{}^{G'}{}_{H'}\\\Box&\equiv\partial_{FG'}\,\partial^{FG'}.\end{aligned} \tag{3.1}$$

Now

$$\tfrac{1}{2}\{\partial_{FG'}\,\partial_D{}^{G'}+\partial_{DG'}\,\partial_F{}^{G'}\}\Psi_{ABC}{}^D=3X_{FDE(A}\Psi_{BC)}{}^{DE}+X_{FDE}{}^D\Psi_{ABC}{}^E \tag{3.2}$$

where

$$X_{ABCD}=\Psi_{ABCD}+2\Lambda\varepsilon_{(A|C|}\varepsilon_{B)D} \tag{3.3}$$

ε_{AB} being Levi–Civita symbols, and (| |) denoting that the symmetrisation excludes all indices between the bars. Hence (3.2) is a second-order expression in the field quantities, and vanishes in the linear approximation. Denoting the operators $\overset{\text{flat}}{\partial}_{AA'}$ by $\hat{\partial}_{AA'}$ we thus have, as the linearised version of (3.1),

$$\begin{aligned}\Box\Psi_{ABCD}&=2\hat{\partial}_{DG'}\,\hat{\partial}_{(C}{}^{H'}\Phi_{AB)}{}^{G'}{}_{H'}\\\Box&=\frac{\partial^2}{\partial t^2}-\bigg(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\bigg)\end{aligned} \tag{3.4}$$

(t, x, y, z) being the Minkowski coordinates of § 2. A solution of (3.4) is immediately given for outgoing waves:

$$\Psi_{ABCD}(P) = \frac{1}{2\pi} \hat{\partial}_{DG'} \hat{\partial}_{(C}{}^{H'} \int_V \frac{\Phi_{AB)}{}^{G'}{}_{H'}}{r^*} \mathrm{d}V \tag{3.5}$$

where r^* is the distance from the field point P to the source points, where $\Phi_{ABA'B'}$ is evaluated at a retarded time $t - r^*$, and where V is any volume which completely encloses all the sources for the field. r^* is related to r (distance of field point from origin 0) via $r^{*2} = r^2 + \xi^2 - 2x^a\xi_a$ ($r^2 = x^a x_a$, $\xi^2 = \xi^a \xi_a$) where source and field points have coordinates (t, ξ^a) and $(t, x^a) = x^\mu$ respectively (lower case Latin letters ranging and summing over 1, 2 and 3). By contracting with suitable combinations of the dyad spinors o^A and ι^A we can obtain the dyad components $\psi_0, \ldots, \psi_4$. Our interest will centre, however, on the component ψ_4, which can be written as

$$\psi_4 = -\frac{2}{\pi} \iota^A \iota^B \iota^C \iota^D \bar{o}^{[E'} \bar{\iota}^{G']} \bar{o}^{[F'} \bar{\iota}^{H']} \hat{\partial}_{DE'} \hat{\partial}_{CF'} \int_V \frac{\Phi_{ABG'H'}}{r^*} \mathrm{d}V \tag{3.6}$$

(the Levi–Civita symbol $\varepsilon^{E'G'} = 2\bar{o}^{[E'}\bar{\iota}^{G']}$ has been used here for lowering spinor indices in (3.5)).

The simplest way of dealing with this equation is to introduce two vectors m^μ, n^μ via the transcription

$$m^\mu = \sigma^\mu{}_{AA'} o^A \bar{\iota}^{A'} \qquad n^\mu = \sigma^\mu{}_{AA'} \iota^A \bar{\iota}^{A'}. \tag{3.7}$$

Using the spin dyad (2.4) we have, in the coordinates x^μ,

$$m^\mu = \frac{1}{\sqrt{2}}(0, q^a) \qquad n^\mu = \tfrac{1}{2}(1, -p^a) \tag{3.8}$$

where

$$\begin{aligned} p^a &= \frac{1}{r}(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ q^a &= \left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}\right) p^a. \end{aligned} \tag{3.9}$$

Since, via the Einstein field equations, $\Phi_{ABA'B'}$ can be written in terms of the energy momentum tensor $T_{\mu\nu}$ and its trace T as

$$\Phi_{ABA'B'} = 4\pi\sigma^\mu{}_{AA'}\sigma^\nu{}_{BB'}(T_{\mu\nu} - \tfrac{1}{4}\eta_{\mu\nu}T) \tag{3.10}$$

with $\eta_{\mu\nu}$ being the Lorentz metric†—diag(1, −1, −1, −1)—we have from (3.6)

$$\psi_4 = 2\{2n^\mu \bar{m}^\nu \bar{m}^\alpha n^\beta - n^\mu n^\nu \bar{m}^\alpha \bar{m}^\beta - \bar{m}^\mu \bar{m}^\nu n^\alpha n^\beta\} \frac{\partial^2 M_{\alpha\beta}}{\partial x^\mu\, \partial x^\nu} \tag{3.11}$$

with

$$M_{\alpha\beta} = \int_V \frac{T_{\alpha\beta}\{t - r^*, \xi^a\}}{r^*} \mathrm{d}V. \tag{3.12}$$

† Note that all raising and lowering of indices is accomplished here by using $\eta_{\mu\nu}$.

Upon performing the coordinate transformation $u = t - r$, $x^{a'} = x^a$, ψ_4 becomes

$$\psi_4 = -\bar{q}^c\bar{q}^d\ddot{M}_{cd} + \bar{q}^c\bar{q}^d\,\partial_c\dot{M}_{0d} + 2\bar{q}^b\bar{q}^{[c}p^{d]}\,\partial_d\dot{M}_{bc} - \tfrac{1}{4}\bar{q}^c\bar{q}^d\,\partial_c\,\partial_d M_{00} + \bar{q}^c\bar{q}^{[d}p^{b]}\,\partial_c\,\partial_d M_{0b}$$
$$+\tfrac{1}{4}(2\bar{q}^a\bar{q}^c p^b p^d - \bar{q}^a\bar{q}^b p^c p^d - p^a p^b\bar{q}^c\bar{q}^d)\,\partial_c\,\partial_d M_{ab} \qquad (3.13)$$
$$\partial_c \equiv \partial/\partial x^c$$

with dots denoting differentiation with respect to u, as before. Now

$$T_{\alpha\beta}\{t - r^*, \xi^a\} = \sum_{n=0}^{\infty}\frac{1}{n!}(r - r^*)^n\frac{\partial^n}{\partial u^n}T_{\alpha\beta}(u, \xi^a) \qquad (3.14)$$

whence $M_{\alpha\beta}$ can be expanded as

$$M_{\alpha\beta} = \frac{N_{\alpha\beta}}{r} + p^a\left(\frac{\dot{N}_{\alpha\beta|a}}{r} + \frac{N_{\alpha\beta|a}}{r^2}\right) + \frac{1}{2}\left[p^a p^b\frac{\ddot{N}_{\alpha\beta|ab}}{r} + (3p^a p^b - \delta^{ab})\left(\frac{\dot{N}_{\alpha\beta|ab}}{r^2} + \frac{N_{\alpha\beta|ab}}{r^3}\right)\right]$$
$$+\frac{1}{6}\left[p^a p^b p^c\frac{\dddot{N}_{\alpha\beta|abc}}{r} + 3p^a(2p^b p^c - \delta^{bc})\frac{\ddot{N}_{\alpha\beta|abc}}{r^2}\right.$$
$$\left. + 3p^a(5p^b p^c - 3\delta^{bc})\left(\frac{\dot{N}_{\alpha\beta|abc}}{r^3} + \frac{N_{\alpha\beta|abc}}{r^4}\right)\right] + \dots \qquad (3.15)$$

where

$$N_{\alpha\beta|c_1c_2\dots c_n}(u) = \int_V T_{\alpha\beta}(u, \xi^a)\xi_{c_1}\xi_{c_2}\dots\xi_{c_n}\,\mathrm{d}V. \qquad (3.16)$$

Let us define quantities $h_{\alpha\beta|c_1c_2\dots c_n}(u)$ by

$$h_{00|c_1c_2\dots c_n} = \frac{N_{00|c_1c_2\dots c_n}}{ma^n}$$
$$h_{0c|c_1c_2\dots c_n} = \frac{N_{0c|c_1c_2\dots c_n}}{ma^{n+1}} \qquad (3.17)$$
$$h_{bc|c_1c_2\dots c_n} = \frac{N_{bc|c_1c_2\dots c_n}}{ma^{n+2}}$$

'm' being a mass parameter (cf § 2), and 'a' a length parameter, for the system. The important point about these quantities (3.17) is that they are dimensionless i.e. they are not affected by any change in units in 'm' or 'a'. They are called (see also Bonnor 1966) the moments of mass, momentum and stress respectively. Substituting (3.17) into (3.15) we see that $M_{\alpha\beta}$, and hence ψ_4, can be written as a singly infinite series in 'a': $\psi_4 = m\sum_{l=0}^{\infty} a^l\overset{\widetilde{(l)}}{\psi_4}$, with $\overset{\widetilde{(l)}}{\psi_4}$ independent of 'm' or 'a'. $\overset{\widetilde{(l)}}{\psi_4}$ will be called the 2^l-pole (or $(1l)$) solution 'related to the sources'. The difference between this 2^l-pole solution and the one given in (2.18) should be stressed; the solution here is defined as the coefficient of a dimensional term, whereas that in (2.18) is defined in terms of coefficients of spin weighted spherical harmonics. The tilde on $\overset{(l)}{\psi_4}$ here is used to avoid confusion with the $(1l)$ solutions in (2.18).

Relations between the moments (3.17) can be obtained via (2.1b), which so far have not been discussed. By linearising these equations we can show that they are equivalent to the familiar conservation equations $\eta^{\alpha\beta}\,\partial T_{\gamma\alpha}/\partial\xi^\beta = 0$ $(\xi^\beta = (t, \xi^a))$. By multiplying them throughout by $\xi_{c_1}\xi_{c_2}\ldots\xi_{c_n}$ and integrating over any volume enclosing the sources we obtain†

$$\dot{h}_{00|c_1c_2\ldots c_n} = -nh_{0(c_1|c_2\ldots c_n)} \tag{3.18a}$$

$$\dot{h}_{0c|c_1c_2\ldots c_n} = -nh_{c(c_1|c_2\ldots c_n)}. \tag{3.18b}$$

The monopole ($l=0$) and dipole ($l=1$) solutions for ψ_4 can easily be shown to vanish. The first non-vanishing contribution to ψ_4 comes from $l=2$. Using (3.18) and the relations $\bar{q}^c\bar{q}_c = \bar{q}^c p_c = 0$, the solution for $\overset{(\widetilde{2})}{\psi_4}$ can be written in terms of spherical harmonics as

$$\overset{(\widetilde{2})}{\psi_4} = \sum_q \left(\frac{\overset{\cdots\cdot}{\overset{(2)}{H_q}}}{6r} + \frac{\overset{\cdots}{\overset{(2)}{H_q}}}{3r^2} + \frac{\overset{\cdot\cdot}{\overset{(2)}{H_q}}}{2r^3} + \frac{\overset{\cdot}{\overset{(2)}{H_q}}}{2r^4} + \frac{\overset{(2)}{H_q}}{4r^5}\right) \bar{\eth}^2(P_2^{|q|}\,\mathrm{e}^{\mathrm{i}q\phi}) \tag{3.19}$$

where

$$\begin{aligned}
\overset{(2)}{H_0} &= \tfrac{1}{2}(K_{11}+K_{22}-2K_{33}) \\
\overset{(2)}{H_1} &= \overline{\overset{(2)}{H}}_{-1} = \tfrac{1}{2}(\mathrm{i}K_{23}-K_{13}) \\
\overset{(2)}{H_2} &= \overline{\overset{(2)}{H}}_{-2} = \tfrac{1}{8}(K_{22}-K_{11}+2\mathrm{i}K_{12})
\end{aligned} \tag{3.20}$$

with

$$K_{ab} = h_{00|ab}. \tag{3.21}$$

This solution is formally identical with $\overset{(2)}{\psi_4}$ in (2.18)

$$\overset{(2)}{\psi_4} = \sum_q \left(\frac{\overset{\cdots\cdot}{\overset{(2)}{h_q}}}{6r} + \frac{\overset{\cdots}{\overset{(2)}{h_q}}}{3r^2} + \frac{\overset{\cdot\cdot}{\overset{(2)}{h_q}}}{2r^3} + \frac{\overset{\cdot}{\overset{(2)}{h_q}}}{2r^4} + \frac{\overset{(2)}{h_q}}{4r^5}\right) \bar{\eth}^2(P_2^{|q|}\,\mathrm{e}^{\mathrm{i}q\phi}) \tag{3.22}$$

and it may seem reasonable to take the $\overset{(2)}{H_q}$ as being equivalent, to within a factor of proportionality, to $\overset{(2)}{h_q}$. This, however, is not the case as a study of $\overset{(\widetilde{3})}{\psi_4}$ shows. The solution is

$$\begin{aligned}
\overset{(\widetilde{3})}{\psi_4} = {}& \sum_q \left(\frac{\overset{\cdots\cdot\cdot}{\overset{(3)}{H_q}}}{90r} + \frac{\overset{\cdots\cdot}{\overset{(3)}{H_q}}}{18r^2} + \frac{\overset{\cdots}{\overset{(3)}{H_q}}}{6r^3} + \frac{\overset{\cdot\cdot}{\overset{(3)}{H_q}}}{3r^4} + \frac{5\overset{\cdot}{\overset{(3)}{H_q}}}{12r^5} + \frac{\overset{(3)}{H_q}}{4r^6}\right) \bar{\eth}^2(P_3^{|q|}\,\mathrm{e}^{\mathrm{i}q\phi}) \\
&+ \sum_q \left(\frac{\overset{\cdots\cdot}{b_q}}{r} + \frac{2\overset{\cdots}{b_q}}{r^2} + \frac{3\ddot{b}_q}{r^3} + \frac{3\dot{b}_q}{r^4} + \frac{3b_q}{2r^5}\right) \bar{\eth}^2(P_2^{|q|}\,\mathrm{e}^{\mathrm{i}q\phi})
\end{aligned} \tag{3.23}$$

† Simply integrating the conservation equations as they stand gives $\dot{h}_{00} = \dot{h}_{0c} = 0$ expressing conservation of mass and linear momentum in the linear approximation.

where

$$\overset{(3)}{H}_0=\tfrac{1}{2}(3K_{113}+3K_{223}-2K_{333})$$
$$\overset{(3)}{H}_1=\overline{\overset{(3)}{H}}_{-1}=\tfrac{1}{8}[K_{111}+K_{122}-4K_{133}+\mathrm{i}(4K_{233}-K_{112}-K_{222})]$$
$$\overset{(3)}{H}_2=\overline{\overset{(3)}{H}}_{-2}=\tfrac{1}{8}(K_{223}-K_{113}+2\mathrm{i}K_{123}) \tag{3.24}$$
$$\overset{(3)}{H}_3=\overline{\overset{(3)}{H}}_{-3}=\tfrac{1}{48}[3K_{122}-K_{111}+\mathrm{i}(3K_{112}-K_{222})]$$

with

$$K_{abc}=h_{00|abc} \tag{3.25}$$

and with $b_q(u)$ being defined in terms of the spin weight -2 quantities $\bar{q}^b\bar{q}^{[c}p^{a]}h_{0a|bc}$ by

$$-\tfrac{4}{3}\bar{q}^b\bar{q}^{[c}p^{a]}h_{0a|bc}=\sum_q b_q\,\bar{\eth}^2(P_2^{|q|}\,\mathrm{e}^{\mathrm{i}q\phi}). \tag{3.26}$$

Thus the terms involving b_q are (when multiplied by a^3) also a part of (3.22), whereas the set of terms involving $\overset{(3)}{H}_q$ is formally identical to the solution $\overset{(3)}{\psi}_4$ of (2.18).

We might expect that were we to calculate $a^4\overset{\widetilde{(4)}}{\psi}_4$ we would find a contribution from that solution to $\overset{(2)}{\psi}_4$ also. In fact, we might expect that all (or, in particular cases, an infinite subset of) $a^l\overset{\widetilde{(l)}}{\psi}_4$ $(l\geqslant 2)$ contribute to $\overset{(2)}{\psi}_4$. Thus it seems that $\overset{(2)}{\psi}_4$ does not, in general, have unique dimension, containing contributions from terms of dimensions ma^l $(l\geqslant 2)$. Likewise it seems that $\overset{(3)}{\psi}_4$ contains contributions from $a^l\overset{\widetilde{(l)}}{\psi}_4$ $(l\geqslant 3)$ and so on. If this is the case it would seem appropriate to describe the $\overset{(l)}{\psi}_4$ of (2.18), and in particular the moments $\overset{(l)}{h}_q(u)$ of those solutions, as being 'dimensionally mixed'.

These statements are merely conjecture at this stage. The calculations required to obtain $\overset{\widetilde{(2)}}{\psi}_4$ and $\overset{\widetilde{(3)}}{\psi}_4$ are long and involved; further solutions would be even more complicated. To facilitate further discussion we therefore turn to a study of the asymptotic shear of congruences of null geodesics—σ^0. It is found, in both linearised and exact theories, that

$$-\ddot{\sigma}^0=\text{coefficient of } r^{-1} \text{ in } \psi_4. \tag{3.27}$$

$\dot{\sigma}^0$ is the Bondi–Sachs news function, which governs the dynamic evolution of the complete field.

It is clear that the only term of (3.13) which will give rise to terms of $\mathrm{O}(r^{-1})$ in an expansion of ψ_4 in inverse powers of r is the first term on the right-hand side of the equation. From (3.13), (3.15)–(3.17) and (3.27) we easily obtain†

$$\sigma^0\equiv m\sum_{l=2}^{\infty}a^l\overset{\widetilde{(l)}}{\sigma}{}^0=mq^aq^b\Big(a^2h_{ab}+a^3p^{c_1}\dot{h}_{ab|c_1}+\frac{a^4}{2!}p^{c_1}p^{c_2}\ddot{h}_{ab|c_1c_2}$$
$$+\ldots+\frac{a^{t+2}}{t!}p^{c_1}p^{c_2}\ldots p^{c_t}\frac{\mathrm{d}^t}{\mathrm{d}u^t}h_{ab|c_1c_2\ldots c_t}+\ldots\Big). \tag{3.28}$$

† Since we are assuming that the source is initially stationary, the functions of integration which arise when we integrate (3.28) twice with respect to u vanish.

Let us restrict ourselves, for simplicity, to axisymmetric radiating systems. A long calculation based on (3.28) yields

$$\overset{(l)}{\widetilde{\sigma}}{}^{0} = \frac{1}{(l-2)!} \frac{\mathrm{d}^{l-2}}{\mathrm{d}u^{l-2}} X_{l-2} \qquad (l \geqslant 2) \tag{3.29}$$

with

$$\begin{aligned} X_n(u, \theta, \phi) = \sum_{\substack{C,D \\ C+D=n \\ D\,\text{even}}} \binom{n}{C} \cos^C \theta \sin^D \theta \bigg[\left(\frac{D \cos^2 \theta - \sin^2 \theta}{D+1} \right) \alpha_{CD}^{(11,2,1)} \\ - \frac{(D + \sin^2 \theta)}{D+1} \alpha_{CD}^{(22,2,-1)} + \sin^2 \theta\, \alpha_{CD}^{(33,0,1)} + \frac{2\mathrm{i}D \cos \theta}{D+1} \alpha_{CD}^{(12,2,0)} \bigg] \\ -2 \sum_{\substack{C,D \\ C+D=n \\ D\,\text{odd}}} \binom{n}{C} \cos^C \theta \sin^{D+1} \theta [\cos \theta\, \alpha_{CD}^{(13,1,1)} + \mathrm{i}\alpha_{CD}^{(23,1,0)}] \qquad (n \geqslant 0) \end{aligned} \tag{3.30}$$

where

$$\alpha_{CD}^{(ef,g,h)} = \frac{1}{ma^{n+2}} \int_0^{2\pi} \int_S T'_{ef}(u, R, z) \cos^{D+g}\phi\, z^C R^{D+h}\, \mathrm{d}S\, \mathrm{d}\phi. \tag{3.31}$$

In (3.31) the coordinates (R, ϕ, z) are cylindrical polars, T'_{ef} are the space components of the energy momentum tensor in these coordinates, and S is any surface $g(R, z) = 0$ enclosing the sources ($\mathrm{d}S = \mathrm{d}R\, \mathrm{d}z$).

A simple but somewhat artificial source model shows quite clearly the dimensional mixing. This model has already been considered by Bonnor (1959) and Rotenberg (1964). We have two particles A, B, each of mass $\frac{1}{2}m$, oscillating symmetrically in a straight line AB about their centre of mass 0, taken to be the origin of a rectangular coordinate system $0xyz$. The coordinates of A, B at time u are taken to be $(0, 0, a_1(u))$ and $(0, 0, -a_1(u))$ respectively, with $0 < a_1 \leqslant \frac{1}{2}a$ for all u. In this system the only non-vanishing components of the energy momentum tensor are T_{33}, T_{30} and T_{00}. This implies that the only non-vanishing $\alpha_{CD}^{(ef,g,h)}$ in (3.30) are $\alpha_{n0}^{(33,0,1)}$, which (putting $T_{33} = T_{33}(z, u)\delta(R)/2\pi R$) are given by

$$\alpha_{n0}^{(33,0,1)}(u) = \int_{-a/2}^{a/2} \frac{z^n T_{33}(z, u)\, \mathrm{d}z}{ma^{n+2}} = h_{33|\overbrace{33\ldots3}^{n\ \text{times}}} \tag{3.32}$$

(the prime above T_{33} is no longer necessary). Using the conservation equations (3.18) we have from (3.29), (3.30)

$$\overset{(l)}{\widetilde{\sigma}}{}^{0} = \frac{1}{l!} \sin^2 \theta \cos^{l-2} \theta \frac{\mathrm{d}^l}{\mathrm{d}u^l} H_l(u) \qquad (l \geqslant 2) \tag{3.33}$$

where

$$H_l(u) \equiv h_{00|\overbrace{33\ldots3}^{l\ \text{times}}} = \int_{-a/2}^{a/2} \frac{z^l T_{00}(z, u)\, \mathrm{d}z}{ma^l} \tag{3.34}$$

whereas the $(1l)$ solution for the r^{-1} part of the ψ_4 given by (2.18) gives, for this two-particle case,

$$\overset{(l)}{\sigma}{}^0 = \frac{-2^l(l-2)!}{(2l)!}\frac{\mathrm{d}^l}{\mathrm{d}u^l}\overset{(l)}{h}(u)P_l^2(\cos\theta) \qquad (l \geqslant 2)$$
$$\overset{(l)}{h}(u) \equiv \overset{(l)}{h}_0(u) \tag{3.35}$$

where $-\overset{(\ddot{l})}{\sigma}{}^0$ is the coefficient of r^{-1} in $\overset{(l)}{\psi}_4$. Using

$$\sin^2\theta\cos^m\theta = \begin{cases} c_{2,m}P_2^2 + c_{4,m}P_4^2 + \ldots + c_{m+2,m}P_{m+2}^2 & \text{for } m \text{ even} \\ c_{3,m}P_3^2 + c_{5,m}P_5^2 + \ldots + c_{m+2,m}P_{m+2}^2 & \text{for } m \text{ odd} \end{cases} \tag{3.36}$$

where the coefficients $c_{r,m}$ are non-zero constants†, we easily see that each $a^l\overset{(\widetilde{l})}{\sigma}{}^0$ (l even) contributes to $\overset{(l)}{\sigma}{}^0, \overset{(l-2)}{\sigma}{}^0, \overset{(l-4)}{\sigma}{}^0, \ldots, \overset{(2)}{\sigma}{}^0$ with similar behaviour for the case l odd. Thus the moments $\overset{(l)}{h}(u)$ are due to an infinite number of contributions, each contribution having different dimension. Specifically we have

$$\overset{(l)}{h}(u) = \frac{-(2l)!}{2^l(l-2)!}\sum_{r=0}^{\infty}\frac{a^{l+2r}}{(l+2r)!}c_{l,l+2r-2}\frac{\mathrm{d}^{2r}}{\mathrm{d}u^{2r}}H_{l+2r}(u) \tag{3.37}$$

the $c_{l,l+2r-2}$ being the coefficients of (3.36).

As mentioned previously, this two-particle case is somewhat artificial in that the masses considered are singularities. However, it should be obvious that in more realistic situations, in which most or all of the terms of (3.30) would be brought into play (as opposed to the single term used in the two-particle case) dimensional mixing is assured for all except the most special systems. Indeed it seems possible that *all* radiating systems (but not, it seems, static systems) give rise to the mixing, although this conjecture might be somewhat difficult to test, vitiated as it is by complicated calculations.

4. Energy momentum radiation from compact sources

The paper of Sachs (1962) introduced a certain group—the Bondi–Metzner–Sachs (or BMS) group—as an asymptotic symmetry group of transformations for an asymptotically flat (or, more precisely, 'AF') space–time. Although not a Lie transformation group we may imitate the methods used in Lie group theory to obtain its infinitesimal generators. Sachs, using a somewhat heuristic quantum argument, proceeded to associate with these generators certain integrals. He then identified some of the integrals with the energy momentum content of the space–time‡.

† The $c_{r,m}$ ($m \geqslant 0$, $2 \leqslant r \leqslant m+2$) are given by

$$c_{r,m} = \begin{cases} \dfrac{(2r+1)2^r m!(\frac{1}{2}m+\frac{1}{2}r+1)!}{(\frac{1}{2}m-\frac{1}{2}r+1)!(m+r+3)!} & \text{if } m+r \text{ is even} \\ 0 & \text{if } m+r \text{ is odd.} \end{cases}$$

‡ For further discussion of the generators of the BMS group see Carmeli (1977).

The rate of energy and linear momentum loss from a radiating source is given as the (retarded) time derivative of the momentum 4-vector P_μ, given by

$$P_\mu = (P_0, P_1, P_2, P_3) = \frac{1}{2}\int \dot{\sigma}^0\dot{\bar{\sigma}}^0(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{4.1}$$

where

$$\int f \equiv \int_{-\infty}^{u} \mathrm{d}u \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} f(u, \theta, \phi)\sin\theta\,\mathrm{d}\phi. \tag{4.2}$$

It should be noted that the integrals (4.1) have also been derived by a purely relativistic method by Tamburino and Winicour (1966) using 'flux linkages' for the asymptotically flat space–time.

The six generators $L^{\alpha\beta}$ $(= -L^{\beta\alpha})$ for angular momentum are

$$\begin{aligned} L^{12} &\equiv L_z = \partial/\partial\phi \\ L^{30} &\equiv R_z = \sin\theta\,\partial/\partial\theta + u\cos\theta\,\partial/\partial u \end{aligned} \tag{4.3a}$$

and, putting $L^{\pm} = L^{13} \pm \mathrm{i}L^{23}$, $R^{\pm} = L^{10} \mp \mathrm{i}L^{20}$,

$$\begin{aligned} L^{\pm} &= \mathrm{e}^{\pm\mathrm{i}\phi}(\partial/\partial\theta \pm \mathrm{i}\cot\theta\,\partial/\partial\phi) \\ R^{\pm} &= -\mathrm{e}^{\pm\mathrm{i}\phi}(\cos\theta\,\partial/\partial\theta \pm \mathrm{i}\,\mathrm{cosec}\,\theta\,\partial/\partial\phi - u\sin\theta\,\partial/\partial u) \end{aligned} \tag{4.3b}$$

from which the angular momentum of the system in non-flat space is given by the integrals

$$\begin{aligned} I\{L_z\} &= \frac{1}{2}\int \dot{\sigma}^0 L_z\bar{\gamma} \\ I\{L^{\pm}\} &= \frac{1}{2}\int (\dot{\sigma}^0 L^{\pm}\bar{\gamma} \mp 2\,\mathrm{e}^{\pm\mathrm{i}\phi}\,\mathrm{cosec}\,\theta\,\dot{\sigma}^0\bar{\gamma}) \end{aligned} \tag{4.4}$$

where

$$\gamma(u, \theta, \phi) \equiv \sigma^0(u, \theta, \phi) - \tfrac{1}{2}\sigma^0(-\infty, \theta, \phi) - \tfrac{1}{2}\sigma^0(\infty, \theta, \phi). \tag{4.5}$$

The integrals involving R_z, $R^{\pm}$ do have physical meaning but are not important here.

We now calculate the dominant contributions to these expressions using the multipole solutions for σ^0 obtained in § 3†. In particular, we require the solutions $\overset{(\widetilde{2})}{\sigma}{}^0$ and $\overset{(\widetilde{3})}{\sigma}{}^0$ which, dropping tildes and introducing a $(2\pi)^{-1/2}$ normalising factor, are given by

$$\begin{aligned} \overset{(2)}{\sigma}{}^0 &= \frac{1}{\sqrt{2\pi}}q^a q^b h_{ab} = \frac{1}{2\sqrt{2\pi}}q^a q^b \ddot{K}_{ab} \\ &= \frac{-1}{\sqrt{3\pi}}\sum_{m_0}\frac{\overset{(\ddot{2})}{H}_{m_0\,2}Y_2{}^{m_0}}{A_{2m_0}} \end{aligned} \tag{4.6}$$

† The coordinates used in (4.1), (4.4) are Bondi–Sachs coordinates u, r, θ, ϕ whereas the coordinates u, r, θ, ϕ used in previous sections are related to flat space–time. From the point of view of the calculations to be performed here these sets of coordinates can be considered equivalent.

and

$$\overset{(3)}{\sigma}{}^0=\frac{1}{\sqrt{2\pi}}q^aq^bp^c\dot{h}_{ab|c}$$

$$=\frac{1}{\sqrt{2\pi}}(\tfrac{1}{6}q^aq^bp^c\dddot{K}_{abc}+\tfrac{4}{3}q^aq^{[b}p^{c]}\ddot{h}_{0c|ba})$$

$$=\frac{-1}{3\sqrt{15\pi}}\sum_{m_1}\frac{\dddot{\overset{(3)}{H}}_{m_1}\,{}_2Y_3{}^{m_1}}{A_{3m_1}}+\frac{8}{\sqrt{3\pi}}\sum_{m_2}\frac{\ddot{g}_{m_2}\,{}_2Y_2{}^{m_2}}{A_{2m_2}}. \tag{4.7}$$

In these equations $K_{c_1c_2\ldots c_n}$, $\overset{(2)}{H}_m$ and $\overset{(3)}{H}_m$ are given by (3.20), (3.21), (3.24) and (3.25), and the conservation equations (3.18) have been used. The quantities A_{2m}, A_{3m} are defined in the appendix, as are ${}_2Y_l^m$. The functions $g_m(u)$ are given by

$$\begin{aligned}
g_0&=\tfrac{1}{4}\mathrm{i}(c_4+c_5)=-\bar{g}_0\\
g_1&=\tfrac{1}{24}(\mathrm{i}c_1+c_3)=-\bar{g}_{-1}\\
g_2&=\tfrac{1}{48}(\mathrm{i}(c_4-c_5)+c_2)=-\bar{g}_{-2}
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
c_1&=h_{02|11}+h_{03|23}-h_{01|12}-h_{02|33}\\
c_2&=h_{01|13}+h_{03|22}-h_{02|23}-h_{03|11}\\
c_3&=h_{01|33}+h_{02|12}-h_{01|22}-h_{03|13}\\
c_4&=h_{03|12}-h_{01|23}\qquad c_5=h_{02|13}-h_{03|12}.
\end{aligned} \tag{4.9}$$

Let us return now to (4.1). Differentiating its first component with respect to u gives

$$\frac{\mathrm{d}P_0}{\mathrm{d}u}=\frac{1}{2}\int_S\dot{\sigma}^0\dot{\bar{\sigma}}^0\,\mathrm{d}\Omega=\frac{m^2a^4}{2}\int_S\dot{\overset{(2)}{\sigma}}{}^0\dot{\overset{(2)}{\bar{\sigma}}}{}^0\,\mathrm{d}\Omega+\mathrm{O}(m^2a^5) \tag{4.10}$$

the integrals being taken over the sphere. $\mathrm{O}(m^pa^q)$ $(p\geq 1, q\geq 0)$ denotes a term of the form $\Sigma_{r\geq p,s\geq q}m^ra^s\overset{(rs)}{\alpha}$, each $\overset{(rs)}{\alpha}$ being independent of m or a. Using (4.6) and the orthonormality condition (A5) gives

$$\frac{\mathrm{d}P_0}{\mathrm{d}u}=\tfrac{2}{15}m^2a^4(\dddot{\overset{(2)}{H}}_0{}^2+12\dddot{\overset{(2)}{H}}_1\dddot{\overset{(2)}{H}}_{-1}+48\dddot{\overset{(2)}{H}}_2\dddot{\overset{(2)}{H}}_{-2})+\mathrm{O}(m^2a^5). \tag{4.11}$$

Similarly we can find the contributions of order m^2a^4 for the rate of linear momentum loss. Using the results (A10) we easily find that all three components vanish, establishing that there exists no linear momentum loss at infinity due to quadrupole–quadrupole interaction of a radiating source. The first non-zero contribution to linear momentum loss comes from the quadrupole–octupole interaction. Written out, we have to find

$$\frac{m^2a^5}{2}\int_S\dot{\overset{(2)}{\sigma}}{}^0\dot{\overset{(3)}{\bar{\sigma}}}{}^0(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)\,\mathrm{d}\Omega+\text{complex conjugate}. \tag{4.12}$$

Once again results (A10) help considerably, and we find

$$\left(\frac{\mathrm{d}P_1}{\mathrm{d}u}, \frac{\mathrm{d}P_2}{\mathrm{d}u}\right) = \frac{8}{315} m^2 a^5 (\mathrm{Re}, \mathrm{Im})\{2\dddot{\overset{(2)}{H}}_0 \ddddot{\overset{(3)}{H}}_{-1} + 20\dddot{\overset{(2)}{H}}_1 \ddddot{\overset{(3)}{H}}_{-2}$$
$$+ 120\dddot{\overset{(2)}{H}}_2 \ddddot{\overset{(3)}{H}}_{-3} - \dddot{\overset{(2)}{H}}_{-1} \ddddot{\overset{(3)}{H}}_0 - 4\dddot{\overset{(2)}{H}}_{-2} \ddddot{\overset{(3)}{H}}_1\}$$
$$+ \frac{64}{15} m^2 a^5 (\mathrm{Re}, \mathrm{Im})\{\dddot{\overset{(2)}{H}}_0 \dddot{g}_{-1} + 4\dddot{\overset{(2)}{H}}_{-1} \dddot{g}_{-2}$$
$$- \dddot{\overset{(2)}{H}}_{-1} \dddot{g}_0 - 4\dddot{\overset{(2)}{H}}_{-2} \ddot{g}_1\} + \mathrm{O}(m^2 a^6) \tag{4.13}$$

and

$$\frac{\mathrm{d}P_3}{\mathrm{d}u} = \frac{4}{315} m^2 a^5 \,\mathrm{Re}\{\dddot{\overset{(2)}{H}}_0 \ddddot{\overset{(3)}{H}}_0 + 16\dddot{\overset{(2)}{H}}_1 \ddddot{\overset{(3)}{H}}_{-1}$$
$$+ 80\dddot{\overset{(2)}{H}}_2 \ddddot{\overset{(3)}{H}}_{-2}\} + \frac{128}{15} m^2 a^5 \,\mathrm{Re}\{\dddot{\overset{(2)}{H}}_{-1} \dddot{g}_1 + 8\dddot{\overset{(2)}{H}}_{-2} \ddot{g}_2\} + \mathrm{O}(m^2 a^6). \tag{4.14}$$

Finally we calculate the integrals corresponding to the rate of angular momentum loss. The fact that the source is in motion only for finite time intervals (in the sense described in the introduction) allows us to set the $\gamma(u, \theta, \phi)$ of (4.5) as $\sigma^0(u, \theta, \phi)$. The rate of loss in the z direction is

$$\frac{\mathrm{d}}{\mathrm{d}u} I\{L_z\} = \frac{m^2 a^4}{2} \int_S \dot{\overset{(2)}{\sigma}}{}^0 L_z \overline{\overset{(2)}{\sigma}}{}^0 \,\mathrm{d}\Omega + \mathrm{O}(m^2 a^5)$$
$$= \tfrac{8}{5} m^2 a^4 \,\mathrm{Im}\{\dddot{\overset{(2)}{H}}_1 \ddot{\overset{(2)}{H}}_{-1} + 8\dddot{\overset{(2)}{H}}_2 \ddot{\overset{(2)}{H}}_{-2}\} + \mathrm{O}(m^2 a^5). \tag{4.15}$$

A lengthy calculation gives for the second equation of (4.4)

$$\frac{\mathrm{d}}{\mathrm{d}u} I\{L^{\pm}\} = \frac{8}{15} m^2 a^4 \{9\ddot{\overset{(2)}{H}}_{\pm 1} \dddot{\overset{(2)}{H}}_{\mp 2} + \ddot{\overset{(2)}{H}}_0 \dddot{\overset{(2)}{H}}_{\mp 1} - \dddot{\overset{(2)}{H}}_0 \ddot{\overset{(2)}{H}}_{\mp 1} - 9\dddot{\overset{(2)}{H}}_{\pm 1} \ddot{\overset{(2)}{H}}_{\mp 2}\} + \mathrm{O}(m^2 a^5). \tag{4.16}$$

5. The rotating rod

As a check on the results of the previous section we consider a specific radiating system: that of the rotating rod.

The rod will be supposed to rotate for a finite period, outside of which it is stationary. A mechanism for starting and stopping the motion has been described by Rotenberg (1972). The rod $A_1 0 A_2$ will be taken to be of mass m, length a, and small uniform cross section $\tilde{s}$, and will be taken to coincide with the x axis of a rectangular coordinate system $0xyz$ at time $u = 0$. At this time it is rotating with angular velocity ω in the xy plane, and continues to rotate with this angular velocity until time $u = U$, whence we have $\measuredangle x 0 A_2 = \omega u$ at time u $(0 \leq u \leq U)$. In the linear approximation we take the centres of both mass and rotation as 0. When nonlinear i.e. $\mathrm{O}(m^2)$ terms are taken into account these centres do not coincide, as we shall show shortly. Write $|0A_i| = aK_i$ $(i = 1, 2)$ and consider any point Q on the rod such that $0Q = a\zeta$ $(-K_1 \leq \zeta \leq K_2)$. Let $\rho(\zeta)$ be the volume density at Q, and define quantities $\overset{(n)}{I}$ (the nth moments of the rod about its centre of mass) by

$$\overset{(n)}{I} = a^{n+1}\tilde{s}\int_{-K_1}^{K_2} \zeta^n \rho(\zeta)\,\mathrm{d}\zeta \tag{5.1}$$

which are calculated for the period of constant spin of the rod i.e. for $0 \leqslant u \leqslant U$. The dimensionless moments (up to $h_{ab|cde}$) have been given by Rotenberg (1968) for this case. The moments we require are

$$\begin{aligned}
&h_{00|11} = \frac{\overset{(2)}{I} c^2}{ma^2} \qquad h_{00|12} = \frac{\overset{(2)}{I} sc}{ma^2} \qquad h_{00|22} = \frac{\overset{(2)}{I} s^2}{ma^2} \\
&h_{01|11} = \frac{\omega \overset{(3)}{I} sc^2}{ma^3} = -h_{02|12} \qquad h_{01|12} = \frac{\omega \overset{(3)}{I} s^2 c}{ma^3} = -h_{02|22} \\
&h_{01|22} = \frac{\omega \overset{(3)}{I} s^3}{ma^3} \qquad h_{02|11} = -\frac{\omega \overset{(3)}{I} c^3}{ma^3} \\
&h_{00|111} = \frac{\overset{(3)}{I} c^3}{ma^3} \qquad h_{00|222} = \frac{\overset{(3)}{I} s^3}{ma^3} \\
&h_{00|112} = \frac{\overset{(3)}{I} sc^2}{ma^3} \qquad h_{00|122} = \frac{\overset{(3)}{I} s^2 c}{ma^3}
\end{aligned} \tag{5.2}$$

all other $h_{00|ab}$, $h_{0a|bc}$, $h_{00|abc}$ vanishing, where

$$s = \sin \omega u \qquad c = \cos \omega u. \tag{5.3}$$

We therefore find from (3.20), (3.24) and (4.8)–(4.9) that

$$\begin{aligned}
&\overset{(2)}{H}_0 = \text{constant} \qquad \overset{(2)}{H}_1 = 0 \qquad \overset{(2)}{H}_2 = \frac{\overset{(2)}{I}}{8ma^2}(s+\mathrm{i}c)^2 \\
&\overset{(3)}{H}_0 = \overset{(3)}{H}_2 = 0 \qquad \overset{(3)}{H}_1 = \frac{\overset{(3)}{I}}{8ma^3}(c-\mathrm{i}s) \qquad \overset{(3)}{H}_3 = \frac{\overset{(3)}{I}}{48ma^3}\{c(3s^2-c^2)+\mathrm{i}s(3c^2-s^2)\}
\end{aligned} \tag{5.4}$$

$$g_0 = g_2 = 0 \qquad g_1 = -\frac{\omega \overset{(3)}{I}}{24ma^3}(s+\mathrm{i}c).$$

The rate of energy momentum loss is now easy to compute. Using (4.11) and (4.13)–(4.16) we find that

$$\frac{\mathrm{d}P_0}{\mathrm{d}u} = \frac{32}{5}\overset{(2)}{I}{}^2\omega^6 + \mathrm{O}(m^2a^5) \tag{5.5a}$$

$$\frac{\mathrm{d}P_a}{\mathrm{d}u} = -\frac{464}{105}\overset{(2)}{I}\,\overset{(3)}{I}\,\omega^7(\sin\omega u, -\cos\omega u, 0) + \mathrm{O}(m^2a^6) \tag{5.5b}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}I\{L_z\} = -\frac{32}{5}\overset{(2)}{I}{}^2\omega^5 + \mathrm{O}(m^2a^5) \tag{5.5c}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}I\{L^{\pm}\} = \mathrm{O}(m^2a^5). \tag{5.5d}$$

Result (5.5*a*) is very well known, and (5.5*b*) has been obtained by Rotenberg (1968) using the pseudotensor. He also obtained a different answer for linear momentum loss using a method originally devised by Synge (1960, chap 4). The discrepancy between the results appears to be due to the fact that the quantities in (5.5*b*) give the rate of momentum loss at (null) infinity in the asymptotically flat space–time, and include a contribution from the gravitational field, whereas the losses obtained by Synge's method relate only to the source itself. The contribution from the gravitational field turns out to be 13/116 times the expression for $\mathrm{d}P_a/\mathrm{d}u$ given by (5.5*b*) (see Rotenberg for details). Using either result, it is clear that the linear momentum of the rod varies cyclically, and that the centre of rotation of the rod does not coincide with its centre of mass.

6. Conclusion

The previous study has been that of the linearised theory of spin 2 fields in the region exterior to the sources producing the fields. The two definitions of multipole moment which occur arise firstly via the solution of first-order linear homogeneous differential equations with rather general boundary conditions relating to asymptotic behaviour, and secondly via inhomogeneous wave equations. Although the homogeneous case has been considered in some detail, the inhomogeneous case has received very little attention. In this latter case a certain proportion of the blame for the lack of progress in earlier work can be attributed to the tensor methods used in the analyses, which turned out to be somewhat involved. The more complete analysis in this paper is largely due to the use of the spinor formalism. This bypasses many of the earlier calculational difficulties; in particular, awkward coordinate transformations (see Rotenberg 1964) are obviated.

As mentioned in the introduction, the first definition of multipole moment is useful in describing *general* features of gravitational radiation, whereas the second definition is the one to be employed whenever calculations for *specific* source configurations are required. Thus we see that it is useful, and arguably necessary, to have two 'moment' definitions in (linearised) general relativity. It does not seem in the literature that a clear distinction between the two types of moment has ever been made. This distinction is obviously necessary considering the widely different dimensional behaviour in the two definitions. Indeed, the lack of awareness regarding dimensional behaviour appears to have led to some incorrect results: in one approximation method† used in recent years in gravitational radiation certain solutions were obtained which seem incorrect, essentially because the form of the angular–dimensional coupling, and hence dimensional mixing, seems incorrect.

† Namely, the double parameter approximation method developed by Bonnor (1959) and Bonnor and Rotenberg (1966). For further discussion regarding this point see Willmer (1977, ch 5).

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Appendix

The spin weight s spherical harmonics are defined by

$$ {}_sY_l^m = \begin{cases} \left(\dfrac{(l-s)!}{(l+s)!}\right)^{1/2} \eth^s Y_l^m & 0 \leqslant s \leqslant l \\ (-1)^s \left(\dfrac{(l+s)!}{(l-s)!}\right)^{1/2} \bar{\eth}^{-s} Y_l^m & -l \leqslant s \leqslant 0 \end{cases} \tag{A1} $$

for general integral s, where $\eth$, $\bar{\eth}$ are spin weight raising and lowering operators defined by

$$ \begin{aligned} \eth\eta &= -(\sin\theta)^s \left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\theta}\right)\{(\sin\theta)^{-s}\eta\} \\ \bar{\eth}\eta &= -(\sin\theta)^{-s} \left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}\right)\{(\sin\theta)^{s}\eta\} \end{aligned} \tag{A2} $$

with η of spin weight s (for further details regarding $\eth$, $\bar{\eth}$ and spin weight see, for example, Couch *et al* (1968)). Y_l^m are ordinary spherical harmonics

$$ Y_l^m(\theta, \phi) = A_{lm} P_l^{|m|}(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\phi} \tag{A3} $$

where

$$ \left(\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right)^{1/2} = \begin{cases} (-1)^m A_{lm} & m \geqslant 0 \\ A_{lm} & m < 0 \end{cases} \tag{A4} $$

and where $P_l^{|m|}(\cos\theta)$ are associated Legendre functions. The spin-s spherical harmonics obey the orthonormality condition

$$ \int_S {}_sY_l^m \, {}_s\bar{Y}_{l'}^{m'} \, \mathrm{d}\Omega = \delta_{ll'}\delta_{mm'} \tag{A5} $$

(integration being taken over the sphere), which helps in the evaluation of the integrals

$$ \int_S {}_2Y_l^m \, {}_2\bar{Y}_{l'}^{m'} (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\, \mathrm{d}\Omega \tag{A6} $$

which arise in (4.12). To calculate (A6) we need the expression for the product of spherical harmonics of spin weights s and t:

$$ {}_sY_l^m \cdot {}_tY_k^n = \sum_{j=|l-k|}^{l+k} \left(\frac{(2l+1)(2k+1)}{4\pi(2j+1)}\right)^{1/2} \langle knlm \,|\, klj, m+n\rangle \times \langle k-tl-s \,|\, klj, -s-t\rangle_{s+t} Y_j^{m+n} \qquad \text{(A7)} $$

where

$$ \langle k\beta l\alpha \,|\, klj, \alpha+\beta\rangle = \left(\frac{(l+k-j)!\,(j+k-l)!\,(j+l-k)!\,(2j+1)}{(j+k+l+1)!}\right)^{1/2} \times \sum_p \frac{(-1)^p((k+\beta)!\,(k-\beta)!\,(l+\alpha)!\,(l-\alpha)!\,(j+\alpha+\beta)!\,(j-\alpha-\beta)!)^{1/2}}{p!\,(k+l-j-p)!\,(k-\beta-p)!\,(l+\alpha-p)!\,(j-l+\beta+p)!\,(j-k-\alpha+p)!} \qquad \text{(A8)} $$

are real Clebsch–Gordan coefficients. The conditions to be satisfied in (A8) are $l+k \geqslant j$, $l+j \geqslant k$, $k+j \geqslant l$; otherwise the coefficients are zero. The summation is over those values of p for which the contents of all the factorials are greater than or equal to zero. Using

$$ (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \sqrt{\tfrac{2}{3}\pi}\,((Y_1^{-1} - Y_1^1), \mathrm{i}(Y_1^{-1} + Y_1^1), \sqrt{2}\,Y_1^0) \qquad \text{(A9)} $$

and (A5), (A7), (A8) the integrals of (A6) are

$$ \begin{aligned} \int_S {}_2Y_l^m \cdot {}_2\bar{Y}_{l'}^{m'} \sin\theta(\cos\phi, \sin\phi)\, \mathrm{d}\Omega &= \frac{1}{2l}\left(\frac{(l+2)(l-2)}{(2l+1)(2l-1)}\right)^{1/2} \{[(l-m)(l-m-1)]^{1/2}(1, -\mathrm{i})\delta_{l-1,l'}\delta_{m+1,m'} \\ &\quad - [(l+m)(l+m-1)]^{1/2}(1, \mathrm{i})\delta_{l-1,l'}\delta_{m-1,m'}\} \\ &\quad + \frac{1}{l(l+1)}\{[(l-m)(l+m+1)]^{1/2}(-1, \mathrm{i})\delta_{ll'}\delta_{m+1,m'} \\ &\quad - [(l+m)(l-m+1)]^{1/2}(1, \mathrm{i})\delta_{ll'}\delta_{m-1,m'}\} \\ &\quad + \frac{1}{2(l+1)}\left(\frac{(l+3)(l-1)}{(2l+3)(2l+1)}\right)^{1/2} \\ &\quad \times \{[(l+m+1)(l+m+2)]^{1/2}(-1, \mathrm{i})\delta_{l+1,l'}\delta_{m+1,m'} \\ &\quad + [(l-m+1)(l-m+2)]^{1/2}(1, \mathrm{i})\delta_{l+1,l'}\delta_{m-1,m'}\} \end{aligned} \qquad \text{(A10}a\text{)} $$

and

$$ \begin{aligned} \int_S {}_2Y_l^m \cdot {}_2\bar{Y}_{l'}^{m'} \cos\theta\, \mathrm{d}\Omega &= \frac{1}{l}\left(\frac{(l+m)(l-m)(l+2)(l-2)}{(2l+1)(2l-1)}\right)^{1/2} \delta_{l-1,l'}\delta_{mm'} - \frac{2m}{l(l+1)}\delta_{ll'}\delta_{mm'} \\ &\quad + \frac{1}{l+1}\left(\frac{(l+m+1)(l-m+1)(l+3)(l-1)}{(2l+3)(2l+1)}\right)^{1/2} \delta_{l+1,l'}\delta_{mm'}. \end{aligned} \qquad \text{(A10}b\text{)} $$

References

Bonnor W B 1959 *Phil. Trans. R. Soc.* A **251** 233
—— 1966 in *Perspectives in Geometry and Relativity* ed B Hoffman (Bloomington: Indiana University Press)
Bonnor W B and Rotenberg M A 1966 *Proc. R. Soc.* A **289** 247
Carmeli M 1977 *Group Theory and General Relativity* (New York: McGraw–Hill)
Couch W E, Torrence R J, Janis A I and Newman E T 1968 *J. Math. Phys.* **9** 484
Janis A I and Newman E T 1965 *J. Math. Phys.* **6** 902
Newman E T and Penrose R 1962 *J. Math. Phys.* **3** 566
Penrose R 1960 *Ann. Phys., NY* **10** 171
Pirani F A E 1964 in *Lectures on General Relativity* (Englewood Cliffs NJ: Prentice-Hall)
Rotenberg M A 1964 *PhD Thesis* University of London
—— 1968 *J. Phys. A: Gen. Phys.* **1** 97
—— 1972 *J. Phys. A: Gen. Phys.* **5** 198
Sachs R 1962 *Phys. Rev.* **128** 2851
Synge J L 1960 *Relativity: the General Theory* (Amsterdam: North-Holland)
Tamburino L A and Winicour J H 1966 *Phys. Rev.* **150** 1039
Torrence R J and Janis A I 1967 *J. Math. Phys.* **8** 1355
Willmer D G 1977 *PhD Thesis* University of London